M.U.
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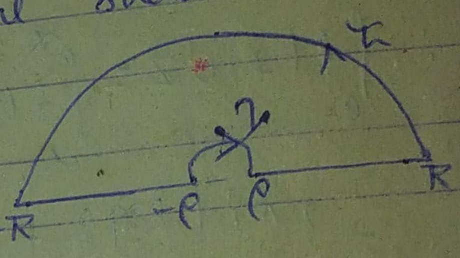
Q.No → By integrating $\frac{e^{iz}}{z}$ around a suitable contour prove that,

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Proof :- We consider the

$$\int_C \frac{e^{iz}}{z}\, dz = \int_C f(z)\, dz$$ where C is contour consisting of the real axis from $-R$ to $-\rho$, the semi-circle γ of small radius ρ, the real axis from ρ to R and the semi-circle Γ of large radius R in the upper half of z plane.



$$\therefore \int_C f(z)\, dz = \int_{-R}^{-\rho} f(x)\, dx + \int_{\gamma} f(z)\, dz + \int_{\rho}^{R} f(x)\, dx + \int_{\Gamma} f(z)\, dz = 0$$

$\therefore f(z)$ has no pole within C since,

$\lim_{z \to \infty} \frac{1}{z} = 0$, by ~~Zordern's~~ Jordan's lemma, we have

$$\lim_{R \to \infty} \int_{\Gamma} \frac{e^{iz}}{z}\, dz = \lim_{R \to \infty} \int_{\Gamma} f(z)\, dz = 0$$

Again, $\lim_{z \to 0} z f(z) = \lim_{z \to 0} z \cdot \frac{e^{iz}}{z}$

$= \lim_{z \to 0} e^{iz} = e^{0} = 1$

$$\therefore \lim_{\epsilon \to 0} \int_{\gamma} f(z)\,dz = i\pi(0-1) = -i\pi$$

$$\therefore \int_{-\infty}^{0} f(x)\,dx - i\pi + \int_{0}^{\infty} f(x)\,dx = 0$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = i\pi$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x}\,dx = i\pi$$

Equating the imaginary part of both sides, we have

$$\therefore \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi \quad \therefore 2\int_{0}^{\infty} \frac{\sin x}{x}\,dx = \pi$$

$$\therefore \int_{0}^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

QNo → Prove that, Integral

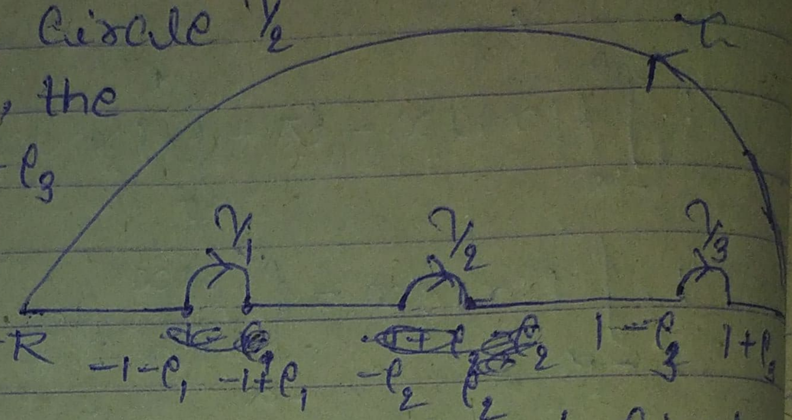
$$\int_{0}^{\infty} \frac{\sin \pi x}{x(1-x^2)}\,dx = \pi.$$

Soln. :→ We Consider the integral

$$\int_{0}^{\infty} \frac{e^{i\pi z}}{z(1-z^2)}\,dz = \int_{C} f(z)\,dz$$

Where, C is the Contour Consisting of real axis from $-R$ to R, the Semi Circle

γ_1 of small radius $+\rho_1$, the real axis from $-R$ $+\rho_1$ to $-\rho_2$, the semi circle γ_2 of small radius ρ_2, the real axis from ρ_2 to $-\rho_3$ the semi circle γ_3 of small radius ρ_3 the real axis from $1+\rho_3$ to arc together with semi circle Γ of large radius R.



$$\therefore \int_C f(z)\,dz = \int_{-R}^{-1-\rho_1} f(x)\,dx + \int_{\gamma_1} f(z)\,dz + \int_{-1+\rho_1}^{-\rho_2} f(x)\,dx$$

$$+ \int_{\gamma_2} f(z)\,dz + \int_{\rho_2}^{1-\rho_3} f(x)\,dx + \int_{\gamma_3} f(z)\,dz + \int_{1+\rho_3}^{R} f(x)\,dx$$

$$+ \int_{\Gamma} f(z)\,dz = 0$$

As $f(z)$ has no singularities within C, clearly,

$$\lim_{z\to\infty} z f(z) = \lim_{z\to\infty} z \frac{e^{i\pi z}}{z(1-z^2)} = 0$$

Hence by ~~Zardern's~~ Jordan's lemma

$$\lim_{R\to 0} \int_{\Gamma} f(z)\,dz = \lim_{R\to\infty} \int_{\Gamma} \frac{e^{i\pi z}}{z(1-z^2)}\,dz = 0$$

$$\lim_{z\to -1} (z+1) f(z) = \lim_{z\to -1} \frac{(z+1)\,e^{i\pi z}}{z(1-z^2)}$$

$$= \lim_{z\to -1} \frac{e^{i\pi z}}{z(1-z)} = \frac{e^{-i\pi}}{-2} = \frac{1}{2}$$

$$\therefore \underset{\rho_1 \to 0}{Lt} \int_{\gamma_1} f(z)\,dz = i(0-\pi)\frac{1}{2} = -\frac{i\pi}{2}$$

Again, $\underset{z \to 0}{Lt}\, z f(z) = \underset{z \to 0}{Lt} \frac{\cancel{z} \cdot e^{i\pi z}}{\cancel{z}(1-z^2)} = 1$

Hence, $\underset{\rho_2 \to 0}{Lt} \int_{\gamma_2} f(z)\,dz = i(0-\pi)\cdot 1 = -\pi i$

Finally, $\underset{z \to 1}{Lt}\,(z-1) f(z) = \underset{z \to 1}{Lt} \frac{(z-1)\,e^{i\pi z}}{z(1-z^2)}$

$$= \underset{z \to 1}{Lt} -\frac{e^{i\pi z}}{z(1+z)} = -\frac{e^{i\pi}}{2} = \frac{1}{2}.$$

Hence, $\underset{\rho_3 \to 0}{Lt} \int_{\gamma_3} f(z)\,dz = i(0-\pi)\frac{1}{2} = -\frac{\pi i}{2}.$

$$\therefore \int_{-\infty}^{-1} f(x)\,dx - \frac{i\pi}{2} + \int_{-1}^{0} f(x)\,dx - i\pi + \int_{0}^{1} f(x)\,dx$$

$$-\frac{i\pi}{2} + \int_{1}^{\infty} f(x)\,dx = 0$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 2i\pi$$

i.e. $\int_{-\infty}^{\infty} \frac{e^{i\pi x}}{x(1-x^2)}\,dx = 2i\pi.$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos\pi x + i\sin\pi x}{x(1-x^2)}\,dx = 2i\pi$$

Equating imaginary Part from both Sides, we have

$$\int_{-\infty}^{\infty} \frac{\sin \pi x}{x(1-x^2)}\, dx = 2\pi.$$

$$\therefore 2\int_{0}^{\infty} \frac{\sin \pi x}{x(1-x^2)}\, dx = 2\pi$$

$$\text{or, } \int_{0}^{\infty} \frac{\sin \pi x}{x(1-x^2)}\, dx = \pi.$$

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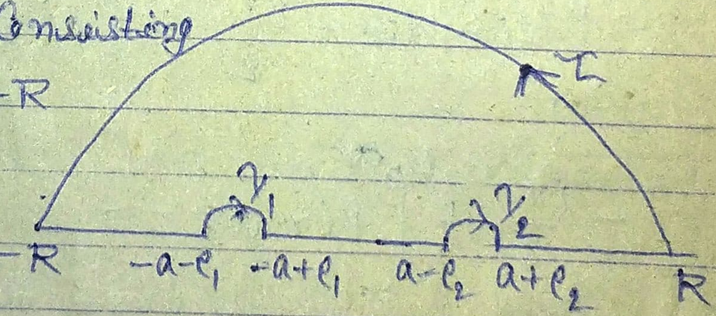
QNo → Prove that if $a > 0$, $\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2}\, dx = \frac{\pi \sin a}{a}$

Soln:- We Consider the integral $\int_C \frac{e^{iz}}{a^2 - z^2} = \int_C f(z)\,dz$

where C is Contour

~~where C is~~ Consisting of real axis from $-R$ to $R, -\epsilon_1$; the semi circle γ_1 of small radius ϵ_1 the real axis $-a+\epsilon_1$ to $a-\epsilon_2$, the semi circle γ_2 of small radius ϵ_2, the real axis from $a+\epsilon_2$ to R together with the semi circle Γ of large radius R, Hence,



$$\int_C f(z)\,dz = \int_{-R}^{-a-\epsilon_1} f(x)\,dx + \int_{\gamma_1} f(z)\,dz + \int_{-a+\epsilon_1}^{a-\epsilon_2} f(x)\,dx$$

$$+ \int f(z)\,dz + \int^{R} f(x)\,dx + \int f(z)\,dz$$

As $f(z)$ has no singularities within C.

Since, $\lim\limits_{R\to 0}\int_{\Gamma} \frac{e^{iz}}{a^2-z^2}\,dz$

$$= \lim_{R\to\infty}\int_{\Gamma} f(z)\,dz = 0$$

$$= \lim_{z\to -a}(z+a)f(z) = \lim_{z\to -a}\frac{(z+a)e^{iz}}{a^2-z^2} = \lim_{z\to -a}\frac{e^{iz}}{a-z}$$

$$= \frac{e^{ia}}{2a}$$

$$\therefore \lim_{\rho_1\to 0}\int_{\gamma_1} f(z)\,dz = -i(0-\pi)\cdot\frac{e^{ia}}{2a} = \frac{i\pi e^{ia}}{2a}$$

Also, $\lim\limits_{z\to a}(z-a)f(z) = \lim\limits_{z\to -a}\frac{(z-a)e^{iz}}{a^2-z^2}$

$$= \lim_{z\to a}\frac{-e^{iz}}{a+z} = -\frac{e^{ia}}{2a}$$

$$\therefore \lim_{\rho_2\to 0}\int_{\gamma_2} f(z)\,dz = -i(0-\pi)\frac{e^{ia}}{2a} = -\frac{i\pi e^{ia}}{2a}$$

$$\therefore \int_{-\infty}^{-a} f(x)\,dx + \frac{-i\pi e^{-ia}}{2a} + \int_{-a}^{a} f(x)\,dx + \frac{i\pi e^{ia}}{2a}$$

$$+ \int_{0}^{\infty} f(x)\,dx = 0$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = -\frac{i\pi}{2a}\left(e^{ia} - e^{-ia}\right)$$

$$= -\frac{i\pi}{2a} \cdot 2i \sin a = \frac{2\pi \sin a}{2a}$$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{ix}}{a^2 - x^2} dx = \int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{a - x^2} = \frac{\pi \sin a}{a}$$

Equating the real Part from both Sides, we have

$$\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2} dx = \frac{\pi \sin a}{a} \quad \text{Proved.}$$